Higher-Order Theory of Homogeneous Plate Flexure

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The need for higher-order models to obtain accurate estimates to transverse shear and normal stresses and strains is well known. In this paper a higher-order shear deformation theory with provision for cubic variation of in-plane displacements and parabolic variation of the normal displacement has been considered for detailed study. Numerical results have been obtained for infinite plate strip, simply supported on opposite edges and infinitely long in the other direction, subjected to sinusoidal static load. Displacements, strains, and stresses are compared with the exact solution. Results by the classical plate theory, Levinson's theory and a higher-order theory based on partial deflections are also included. This study shows that the present model can predict all displacements, strains, and stresses reasonably accurately. Further, it is also noted that the statically equivalent estimates to transverse stresses from classical theory and the corresponding strains also agree closely with the exact solution.

Introduction

HERE we are concerned with the modeling of plates subjected to transverse loading. In the classical theory of thin plates, there is no provision for normal and transverse shear strains. However, the transverse shear and the normal stresses are obtained using the equilibrium equations. Obviously this involves violation of the constitutive law. Shear deformation theories aim at improving the theory by incorporating the transverse shear strains.[1-3] Primarily two approaches are available for this purpose. In the first one, which is attributed to Reissner,[2] stresses are treated as primary variables. In the second approach, attributed to Timoshenko,[1] Mindlin,[3] and others, displacements are treated as primary variables. We consider the second approach here. The displacement field is chosen in a suitable form, and the potential energy principle is used to formulate the governing equations. Strains and stresses are obtained from the displacements using the strain-displacement and constitutive relations. In this approach, it is expedient to obtain some stresses employing the local equilibrium equations rather than the constitutive relations as done in the classical theory. Both these approaches provide possibilities for further development by bringing in higher-order terms.

With the recognition of the possibility of delamination in the laminated fiber-reinforced plastic composites, the requirements of the theory of flexure has undergone a change. It is no longer adequate to be able to predict the in-plane stresses accurately. The transverse shear and the normal stresses and strains become important because of their role in initiating the interlaminar failure. Thus, a new theory with capabilities to predict not only in-plane stresses that can be used to ensure laminate strength but also the transverse shear and normal stresses to facilitate ensuring interlaminar strengths becomes an essential need. Currently there are a number of notable contributions towards this objective appearing in the literature.[6-16]

The purpose of this paper is to examine the displacement approach in some detail from the point of view of obtaining complete information regarding displacements, strains, and stresses uniquely as an essential requirement for a candidate theory for the analysis of composite panels. The literature on this subject is enormous. Broadly, three major approaches can be identified to provide avenues for the construction of efficient models for the flexure of composites panels. In the first one, the displacements are expanded in series form.[5-6] The expressions for displacements in Ref. 5 are intended only for flexure of plates and also satisfy the zero shear strain condition on the surface of the plate, whereas the expressions used in Ref. 6 cover a more general situation and do not satisfy the zero shear condition at the surface of the plate. Retaining only an appropriate number of terms and using the principle of total potential energy or static equilibrium considerations, governing equations can be generated. In the second approach,[8] interlaminar stresses are directly introduced as additional variables and Reissner's principle is adopted. Reference 9 outlines a third modeling possibility wherein statically equivalent stresses obtained from the classical theory are used to estimate the transverse normal stresses and strains. Viewed as an iterative method, this represents the first of a hierarchy of models conceivable with this concept and provides a new possibility of improving approximations in displacement-based models. By and large, the displacement-based modeling has received relatively more attention. In 1981 Murthy[10] formulated a theory for flexure of layered panels retaining up to cubic terms in in-plane displacements and treating the normal displacement as constant in the thickness direction of the panel. Displacement functions were constructed so that the shear strains at the top and bottom surfaces were zero. These functions are the same as those given earlier in Ref. 5 by the present author, in studying the vibrational behavior of the thick plates and by Levinson.[7] Murthy[10] used averaged displacements as basic variables. Bhimaraddi and Stevens[11] and Reddy[12] used actual displacements instead of averaged displacements as primary variables, and variationally consistent equations were given for dynamic analysis by Bhimaraddi and Stevens[11] and by Reddy[12] for the static case. In all these formulations, the chosen displacement field is such that it constrains the transverse shear strains to be zero wherever the displacements are constrained. Therefore, these models may give unacceptable errors at points in plates where displacements are zero such as those along fixed edges. Reference 13 contains an attempt to correct the situation by introducing an additional variable in the form of an additional partial transverse deflection. The transverse deflection is

Received Oct. 6, 1986; revision received Sept. 30, 1987.

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split into two parts, one contributing to bending and the other to transverse shearing so that the later part is available for providing the transverse shear strain at fixed points in the plates. This formulation provides for only constant shear strain near fixed points in the plates, although over the rest of the plate it can give nearly parabolic variations of transverse shear strain.[7] More recently, Lo, Christenson, and Wu[6] and Tarun Kant[14] have studied higher-order models retaining not only cubic terms in in-plane displacements but also a quadratic term in normal displacement. The expressions for displacements considered by these authors do not satisfy the zero shear strain conditions at the surface of the plate. The present model is formulated by retaining cubic terms in in-plane displacements and quadratic terms in the normal displacement in such a manner that zero-shear strain conditions on the surface of the plate are also satisfied.

Any attempt to satisfy the normal stress condition at the plate surfaces will render the expansion for the displacement dependent on the nature of the applied loads. Wang and Dickson[22] gave an interesting theory of beams to various orders wherein the normal stress condition is also incorporated. In order to keep the theory relatively simple, no attempt is made here to satisfy the normal stress condition a priori. However, we will see later that statically equivalent estimates to normal stress are accurate and also satisfy the normal stress condition.

In order to gain a deeper insight and to establish relative merits of various models, the present study is limited to isotropic plate configuration for which it is possible to generate exact solutions. The governing equations and boundary conditions of plates subjected to transverse loading are given by retaining cubic variation of in-plane displacements and parabolic variation of the normal displacement across the depth of the plate. Numerical results have been obtained for infinite plate strip simply supported on two opposite edges subjected to a sinusoidal load. Results are compared with the exact solution. A comparison of results of the present theory with results obtained by other theories available in the literature is also included.

Formulation

Figure 1 shows a typical plate subjected to normal loading. The material of the plate is assumed to be isotropic, and the deformations are small. The load is assumed to have been applied at top and bottom surfaces equally such that

$$(\sigma_z)_{z=\pm h} = \pm Q \tag{1}$$

As indicated earlier, a two-dimensional model of such a problem can be attempted by expanding the displacements in the series form as

$$\bar{U}(x,y,z) = u_1(x,y) + \xi^3 u_3(x,y) + \xi^5 u_5(x,y) + \ldots$$
$$\bar{V}(x,y,z) = v_1(x,y) + \xi^3 v_3(x,y) + \xi^5 v_5(x,y) + \ldots$$
$$\bar{W}(x,y,z) = w_0(x,y) + \xi^2 w_2(x,y) + \xi^4 w_4(x,y) + \ldots \tag{2}$$

where

$$\xi = z/h$$

Symmetric terms in $\bar{U}$ and $\bar{V}$ and antisymmetric terms in $\bar{W}$ are omitted since they are not relevant in the bending problem of isotropic plates, and u_1 and v_1 are identified as

$$u_1 = -hw_{0,x}$$
$$v_1 = -hw_{0,y} \tag{3}$$

to accommodate Kirchhoff's theory as a special case. Furthermore, the series is modified, as shown here, to incorporate zero shear strain condition at the top and bottom surfaces of the plate

$$\bar{U} = -zw_{0,x} - \sum_{n=3,5,\ldots} p^{(n)} u_n$$
$$\bar{V} = -zw_{0,y} - \sum_{n=3,5,\ldots} p^{(n)} v_n$$
$$\bar{W} = w_0 + \sum_{n=2,4,\ldots} p^{(n)} w_n \tag{4}$$

where

$$p^{(n)} = \xi[1 - (\xi^{n-1}/n)]$$
$$q^{(n)} = (1 - \xi^n) \tag{5}$$

From Eq. (4), it may be noted that when $u_n = v_n = w_n = 0$, the theory corresponds to Kirchhoff's plate theory, whereas with $w_n = 0$, $n = 2,4,\ldots$, $u_n = v_n = 0$, $n = 5,7,\ldots$ it corresponds to Levinson's theory.

The strain-displacement relationships are

$$\epsilon_x = -zw_{0,xx} - \Sigma\, p^{(n)} u_{n,x}$$
$$\epsilon_y = -zw_{0,yy} - \Sigma\, p^{(n)} v_{n,y}$$
$$\epsilon_z = \Sigma\, q^{(n)}_{,z} w_n$$
$$\epsilon_{xy} = -2zw_{0,xy} - \Sigma\, p^{(n)}(u_{n,y} + v_{n,x})$$
$$\epsilon_{xz} = \Sigma\, q^{(n)}[-u/h + w_{n,x}]$$
$$\epsilon_{yz} = \Sigma\, q^{(n)}[-v/h + w_{n,y}] \tag{6}$$

Equation (6) clearly indicates that it is necessary to retain at least the w_2 term in the expansion for $\bar{W}$, Eq. (4), in order to retain the contribution of the normal displacement to the shear strains, otherwise the transverse shear strains will be restricted to zero at points in the plate where the in-plane displacements are zero, such as those along the fixed edges in a plate. In the present model, the w_2 term is associated with quadratic variation of normal displacement. A simpler model with w_2 treated as functions of x and y only has been studied in Refs. (15) and (16). In the present study, we shall consider a model retaining a single term in each of the three displacements in addition to the Kirchhoff's variable w_0; for the sake of convenience, the notation is changed at this stage as follows:

$$w_0 = W,\ u_3 = u,\ v_3 = v,\ w_2 = w$$

and

$$p^{(3)} = p,\ q^{(2)} = q \tag{7}$$

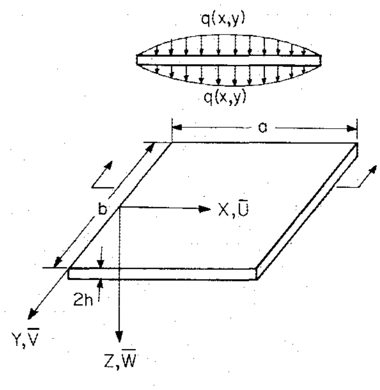

Fig. 1 Geometry and coordinate system.

so that

$$\bar{U} = -zW_{,x} - pu$$
$$\bar{V} = -zW_{,y} - pv$$
$$\bar{W} = W + qw \quad (8)$$

The constitutive relationship can be written in the form

$$\sigma_i = \lambda[\mu e + (1-\mu)\epsilon_i], \quad i = x,y,z$$
$$\sigma_{ij} = [\lambda(1-\mu)/2]\epsilon_{ij}, \quad i,j = x,y,z,\ i \neq j \quad (9)$$
$$e = \epsilon_x + \epsilon_y + \epsilon_z$$
$$\lambda = \frac{E(1-\upsilon)}{(1+\upsilon)(1-2\upsilon)}$$
$$\mu = \upsilon/1 - \upsilon$$
$$G = E/2\,(1+\upsilon) = \lambda g \quad (10)$$

where E is the Young's modulus, G the shear modulus and υ is the Poisson's ratio.

Using the principle of minimum total potential energy in the form $\delta(S - W) = 0$, where S and W are the strain energy and the work potential functions respectively, the governing equations and the boundary conditions may be derived, in terms of the stress resultants, as

$$M_{x,xx} + 2M_{xy,xy} + M_{y,yy} = -2Q$$
$$H_{x,x} + H_{xy,y} = V_x/h$$
$$H_{xy,x} + H_{y,y} = V_y/h$$
$$V_{x,x} + V_{y,y} = H_z \quad (11)$$

Boundary conditions of the plates subjected to transverse loading only and no applied loads along the edges may be written as follows along edges where $x =$ constant, either

$$W = 0 \quad \text{or} \quad M_{x,x} + 2M_{xy,y} = 0$$
$$W_{,x} = 0 \quad \text{or} \quad M_x = 0$$
$$u = 0 \quad \text{or} \quad H_x = 0$$
$$v = 0 \quad \text{or} \quad H_{xy} = 0$$
$$w = 0 \quad \text{or} \quad V_x = 0 \quad (12a)$$

along edges where $y =$ constant, either

$$W = 0 \quad \text{or} \quad M_{y,y} + 2M_{xy,x} = 0$$
$$W_{,y} = 0 \quad \text{or} \quad M_y = 0$$
$$u = 0 \quad \text{or} \quad H_{xy} = 0$$
$$v = 0 \quad \text{or} \quad H_y = 0$$
$$w = 0 \quad \text{or} \quad V_y = 0 \quad (12b)$$

at corners, that is at points where $x =$ constant and $y =$ constant will intersect.

$$W = 0 \quad \text{or} \quad M_{xy} = 0 \quad (12c)$$

When the plate is subjected to edge stress, the boundary conditions can easily be modified by including appropriate stress resultants of the applied stresses in the boundary conditions.

For isotropic plates, these resultants may be written in terms of displacements as

$$M_x = \int_{-h}^{h} \sigma_x z\, dz = -\lambda[I_1(W_{,xx} + \mu W_{,yy}) + I_2(u_{,x} + \mu v_{,y}) + \mu S_2 w]$$

$$M_y = \int_{-h}^{h} \sigma_y z\, dz = -\lambda[I_1(W_{,yy} + \mu W_{,xx}) + I_2(v_{,y} + \mu u_{,x}) + \mu S_2 w]$$

$$M_{xy} = \int_{-h}^{h} \sigma_{xy} z\, dz = -\lambda g[2I_1 W_{,xy} + I_2(u_{,y} + v_{,x})]$$

$$H_x = \int_{-h}^{h} \sigma_x p\, dz = -\lambda[I_2(W_{,xx} + \mu W_{,yy}) + I_3(u_{,x} + \mu v_{,y}) + \mu S_1 w]$$

$$H_y = \int_{-h}^{h} \sigma_y p\, dz = -\lambda[I_2(W_{,yy} + \mu W_{,xx}) + I_3(v_{,y} + \mu u_{,x}) + \mu S_1 w]$$

$$H_{xy} = \int_{-h}^{h} \sigma_{xy} p\, dz = -\lambda g[2I_2 W_{,xy} + I_3(u_{,y} + v_{,x})]$$

$$V_x = \int_{-h}^{h} \sigma_x p'\, dz = \lambda g I_4(-u + w_{,x})$$

$$V_y = \int_{-h}^{h} \sigma_y p'\, dz = \lambda g I_4(-v + w_{,y})$$

$$H_z = \int_{-h}^{h} \sigma_z p''\, dz = \lambda[S_3 w + \mu S_2(W_{,xx} + W_{,yy}) + \mu S_1(u_{,x} + v_{,y})] \quad (13)$$

Using Eq. (13) in Eqs. (11) and (12), the governing equations and boundary conditions will become

$$I_1\nabla^4 W + I_2\nabla^2(u_{,x} + v_{,y}) + \mu S_2 \nabla^2 w = 2Q/\lambda$$

$$I_2\nabla^2 W_{,x} + I_3\left[\frac{1-\mu}{2}u_{,yy} + u_{,xx} + \frac{1+\mu}{2}v_{,xy}\right] + \left[\mu S_1 + \frac{1-\mu}{2}I_4\right]w_{,x} - \frac{1-\mu}{2}I_4 u = 0$$

$$I_2\nabla^2 W_{,y} + I_3\left[\frac{1-\mu}{2}v_{,xx} + v_{,yy} + \frac{1+\mu}{2}u_{,xy}\right] + \left[\mu S_1 + \frac{1-\mu}{2}I_4\right]w_{,y} - \frac{1-\mu}{2}I_4 v = 0$$

$$\frac{1-\mu}{2}I_4\nabla^2 w - \left[\frac{1-\mu}{2}I_4 + \mu S_1\right](u_{,x} - v_{,y}) - S_3 w - \mu S_2 \nabla^2 W = 0 \quad (14)$$

At $x =$ constant, either

$$W = 0 \text{ or } I_1(W_{,xxx} + (2-\mu)W_{,xyy}) + I_2(u_{,xx} + (1-\mu)u_{,yy}) + I_2 v_{,xy} + \mu S_2 w_{,x} = 0$$

$$W_{,x} = 0 \quad \text{or} \quad I_1(W_{,xx} + \mu W_{,yy}) + I_3(u_{,x} + \mu v_{,y}) + \mu S_2 w = 0$$

$$u = 0 \quad \text{or} \quad I_2(W_{,xx} + \mu W_{,yy}) + I_3(u_{,x} + \mu v_{,y}) + \mu S_1 w = 0$$

$$v = 0 \quad \text{or} \quad 2I_2 W_{,xy} + I_3(u_{,y} + v_{,x}) = 0$$

$$w = 0 \quad \text{or} \quad I_4(-u + w_{,x}) = 0 \quad (15a)$$

At y = constant, either

$$W = 0 \text{ or } I_1(W_{,yyy} + (2-\mu)W_{,xxy}) + I_2(v_{,yy} + (1-\mu)v_{,xx}) + I_2 u_{,xy} + \mu S_2 w_{,y} = 0$$

$$W_{,y} = 0 \quad \text{or} \quad I_1(W_{,yy} + \mu W_{,xx}) + I_2(v_{,y} + \mu u_{,x}) + \mu S_2 w = 0$$

$$u = 0 \quad \text{or} \quad 2I_2 W_{,xy} + I_3(u_{,y} + v_{,x}) = 0$$

$$v = 0 \quad \text{or} \quad I_2(W_{,yy} + \mu W_{,xx}) + I_3(v_{,y} + \mu u_{,x}) + \mu S_1 w = 0$$

$$w = 0 \quad \text{or} \quad I_4(-v + w_{,y}) = 0 \tag{15b}$$

and at corners, either

$$W = 0 \quad \text{or} \quad 2I_1 W_{,xy} + I_2(u_{,y} + v_{,x}) = 0 \tag{15c}$$

where

$$I_0 = \int_{-h}^{h} q \, dz = \frac{4h}{3}, \qquad I_4 = \int_{-h}^{h} q^2 \, dz = \frac{16}{15} h$$

$$I_1 = \int_{-h}^{h} z^2 \, dz = \frac{2h^3}{3}, \qquad S_1 = \int_{-h}^{h} pq' \, dz = \frac{16}{15}$$

$$I_2 = \int_{-h}^{h} pz \, dz = \frac{8}{15} h^2, \qquad S_2 = \int_{-h}^{h} q'z \, dz = \frac{4h}{3}$$

$$I_3 = \int_{-h}^{h} p^2 \, dz = \frac{136}{315} h, \qquad S = \int_{-h}^{h} q'^2 \, dz = \frac{8}{3h} \tag{16}$$

Results and Discussion

An isotropic rectangular plate with sides a and b and thickness $2h$ subjected to transverse loading is considered for detailed numerical study. The plate is simply supported at $x = o,a$ and is free at $y = \pm b/2$. The loading is taken in the form

$$(\sigma_z)_{z=\pm h} = \pm q_0 \sin(\pi x/a)$$

Following Refs. 19 and 20, an exact solution has been worked out for the case $b \to \infty$, and this solution is used for comparative assessment. In the numerical study, a is taken to be unity and $b \to \infty$ so that $\bar{V}$ and all derivatives with respect to y can be omitted.

A closed form solution for this case by the present theory can be obtained by starting with expressions for displacements in the form

$$W = A \sin(\pi x/a)$$
$$w = B \sin(\pi x/a)$$
$$u = C \cos(\pi x/a)$$
$$v = 0 \tag{17}$$

where A, B, and C are constants to be evaluated directly from the three algebraic equations that follow after substituting Eq. (17) in the governing equations generated from Eq. (13) by setting v, and all derivatives with respect to y, equal to zero. For this case $\lambda = E/1 - \upsilon^2$ and $\mu = \upsilon$. The solution by the present theory, abbreviated hereafter as PT, may be written as

$$W^* = \frac{\bar{W}}{W_0 \sin(\pi x/a)} = 10\alpha^2(1-\upsilon^2)[A + (1-\xi^2)B]$$

$$U^* = \frac{\bar{U}}{W_0 \cos(\pi x/a)} = -10\alpha^2(1-\upsilon^2)[\alpha\xi A + \xi(1-\xi^2/3)C]$$

$$\epsilon_x^* = \frac{\epsilon_x}{\epsilon_0 \sin(\pi x/a)} = 10\alpha(1-\upsilon^2)[\alpha\xi A + \xi(1-\xi^2/3)C]$$

$$\epsilon_z^* = \frac{\epsilon_z}{\epsilon_0 \sin(\pi x/a)} = -20\xi(1-\upsilon^2)B$$

$$\epsilon_{xz}^* = \frac{\epsilon_{xz}}{\epsilon_0 \cos(\pi x/a)} = 10(1-\upsilon^2)(1-\xi^2)(B\alpha - C)$$

$$\sigma_x^* = \frac{\sigma_x}{q_0 \sin(\pi x/a)} = \frac{30}{\alpha^2}[\alpha^2\xi A - 2\xi\upsilon B + \alpha\xi(1-\xi^2/3)C]$$

$$\sigma_z^* = \frac{\sigma_z}{q_0 \sin(\pi x/a)} = \frac{15\{\upsilon\alpha^2 + 210(1-\upsilon^2)\}\xi(1-\xi^2/3)}{4\{\alpha^4 + 5(1+\upsilon)\alpha^2 + 525(1-\upsilon^2)\}}$$

$$\sigma_{xz}^* = \frac{\sigma_{xz}}{q_0 \cos(\pi x/a)} = \frac{15(1-\upsilon)}{\alpha^2}(1-\xi^2)(B\alpha - C)$$

where

$$W_0 = \frac{2q_0 a^4}{E I \pi^4}, \quad \epsilon_0 = \frac{3q_0}{\alpha^2 E} \quad \text{and} \quad \alpha = \frac{\pi h}{a}$$

$$A = \frac{17(1-\upsilon)\alpha^4 - 85(\upsilon - 1)\alpha^2 + 105(1-\upsilon)}{2(1-\upsilon)\alpha^2\{\alpha^4 + 5(1+\upsilon)\alpha^2 + 525(1-\upsilon^2)\}}$$

$$B = \frac{1}{4(1-\upsilon)} \frac{(43\upsilon - 42)\alpha^2 + 105\upsilon(1-\upsilon)}{\{\alpha^4 + 5(1+\upsilon)\alpha^2 + 525(1-\upsilon^2)\}}$$

$$C = \frac{-21\alpha\{2(1-\upsilon)\alpha^2 - 5\upsilon(1+\upsilon) + 10\}}{4(1-\upsilon)\{\alpha^4 + 5(1+\upsilon)\alpha^2 + 525(1-\upsilon^2)\}} \tag{18}$$

Levinson's theory (LT) based on cubic in-plane displacements, higher-order theory based on cubic in-plane displacements and partial deflections[18] (HTPD), and the classical theory of plates (CPT) are considered here for the comparative study. In CPT, the shear and normal strains are zero, and the estimates for σ_{xz} and σ_z are obtained by integrating the equilibrium equations (see the Appendix), whereas in LT and HTPD the shear stress is obtained using the constitutive law and the normal stress by integrating the equilibrium equations. In the present theory, all stresses are obtained using the constitutive law. Estimates to σ_z obtained by integrating equilibrium equation in PT are also given.

Table 1 Comparison of the normal stress σ_x^*

a/h	ξ	CPT	LT	HTPD	PT	Exact
100	0.2	607.93	667.89	667.89	607.79	607.80
	0.6	1823.78	2003.95	2003.95	1823.60	1823.60
	1.0	3039.64	3340.84	3340.84	3040.15	3040.00
25	0.2	38.00	41.59	41.59	37.85	37.88
	0.6	113.99	125.05	125.05	113.80	113.84
	1.0	189.98	209.34	209.34	190.48	190.38
10	0.2	6.08	6.52	6.52	5.94	5.97
	0.6	18.24	19.84	19.84	18.05	18.09
	1.0	30.40	33.97	33.97	30.91	30.80

Table 2 Comparison of the normal stress σ_z^*

a/h	ξ	CPT* and LT*	HTPD*	PT*	PT	Exact
100	0.2	0.2960	0.2960	0.2960	0.1975	0.2960
	0.6	0.7920	0.7920	0.7920	0.6654	0.7920
	1.0	1.0000	1.0000	1.0000	1.3517	1.0000
25	0.2	0.2960	0.2961	0.2960	0.1974	0.2960
	0.6	0.7920	0.7918	0.7919	0.6651	0.7919
	1.0	1.0000	1.0000	0.9998	1.3518	1.0000
10	0.2	0.2960	0.2953	0.2956	0.1966	1.2960
	0.6	0.7920	0.7906	0.7910	0.6639	0.7908
	1.0	1.0000	1.0000	0.9988	1.3526	1.0000

Tables 1–8 give a detailed comparison of displacements, strain, and stress parameters for three values of the length to thickness ratios. The symbol * indicates statically equivalent estimates.

Comparison of results indicate the following:

1) The PT gives all the displacement, strain, and stress parameters except the normal stress σ_z quite accurately. The normal stress condition is not explicitly satisfied a priori. The error in the normal stress parameter is about 35% (see Table 2). Statically equivalent estimate to σ_z is, of course, very accurate. Statically equivalent estimates for σ_{xz} in the case of CPT and σ_z in the case of CPT, LT, and HTPD also agree closely with the exact solution (see Tables 2 and 3).

2) Estimates to the stress σ_x by LT and HTPD are slightly inferior when compared to CPT. The PT gives the best correlation with the exact, and CPT follows it closely (see Table 1).

3) The shear stresses estimated by LT, HTPD, and PT using the constitutive law and the statically equivalent shear stresses obtainable from CPT are all too close to exact solution (see Table 3).

4) PT gives the best approximation of the normal deflection. LT and HTPD also predict the normal deflection well, whereas CPT has the largest error (see Table 4).

5) The in-plane displacement is predicted quite well by all the theories. PT, CPT, HTPD, and LT are close to the exact in that order (see Table 5).

6) The strain ϵ_x is obtainable using the constitutive relations in all cases and agree closely with the exact solution (see Table 6).

7) In the PT, transverse normal strain ϵ_z can be obtained using the constitutive law, and this agrees reasonably well with the exact solution (see Table 7).

8) The transverse shear strain ϵ_{xz} can be obtained using the constitutive law in LT, HTPD, and PT, and all of them agree closely with exact solution. HTPD leaves errors on the plate surfaces as should be expected (see Table 8).

9) Strains ϵ_x, ϵ_z, and ϵ_{xz} obtained from statically equivalent stresses predicted in CPT also agree very closely with the exact solution (see Table 9).

Table 3 Comparison of shear stress σ^*_{xz}

a/h	ξ	CPT*	LT	HTPD	PT	Exact
100	0.0	47.7465	47.7478	47.7454	47.7460	47.744
	0.2	45.8367	45.8369	45.8356	45.8361	45.834
	0.6	30.5578	30.5579	30.5579	30.5574	30.558
25	0.0	11.9366	11.9366	11.9320	11.9344	11.930
	0.2	11.4521	11.4592	11.4551	11.4570	11.454
	0.6	7.6394	7.6394	7.6398	7.6380	7.643
10	0.0	4.7747	4.7747	4.7631	4.7689	4.759
	0.2	4.5837	4.5837	4.5735	4.5781	4.572
	0.6	3.0558	3.0558	3.0567	3.0521	3.064

Table 4 Comparison of deflection W^*

a/h	ξ	LT	HTPD	PT	Exact
100	0.0	1.0010	1.0010	1.0009	1.0009
	1.0	1.0010	1.0010	1.0008	1.0008
25	0.0	1.0164	1.0164	1.0150	1.0150
	1.0	1.0164	1.0164	1.0127	1.0127
10	0.0	1.1026	1.1025	1.0933	1.0932
	1.0	1.1026	1.1025	1.0804	1.0805

Table 5 Comparison of displacement U^* at $\xi = 1$

a/h	CPT	LT	HTPD	PT	Exact
100	−0.03142	−0.03142	−0.03142	−0.03142	−0.031416
25	−0.12566	−0.12601	−0.12601	−0.12573	−0.12573
10	−0.31416	−0.31953	−0.31949	−0.31527	−0.31526

Table 6 Comparison of axial strain ϵ^*_x at $\xi = 1$

a/h	CPT	LT	HTPD	PT	Elasticity
100	1.0	1.00017	1.00017	1.00003	1.00003
25	1.0	1.0027	1.0027	1.0005	1.0005
10	1.0	1.0171	1.0170	1.0035	1.0035

Table 7 Comparison of normal strain ϵ^*_z

a/h	ξ	CPT, LT, HTPD	PT	Elasticity
100	0.2	0	−0.0600	−0.0600
	0.6	0	−0.1780	−0.1797
	1.0	0	−0.2996	−0.2997
25	0.2	0	−0.0587	−0.0583
	0.6	0	−0.1762	−0.1756
	1.0	0	−0.2937	−0.2954
10	0.2	0	−0.0521	−0.0493
	0.6	0	−0.1563	−0.1525
	1.0	0	−0.2606	−0.2711

Table 8 Comparison of shearing strain ϵ^*_{zx}

a/h	ξ	CPT	LT	HTPD	PT	Elasticity
100	0.0	0	0.0408	0.0408	0.0408	0.0408
	0.4	0	0.0343	0.0343	0.0343	0.0343
	0.8	0	0.0147	0.0147	0.0147	0.0147
25	0.0	0	0.1634	0.1633	0.1633	0.1633
	0.4	0	0.1372	0.1372	0.1372	0.1372
	0.8	0	0.0588	0.0589	0.0588	0.0589
10	0.0	0	0.4084	0.4074	0.4079	0.4071
	0.4	0	0.3431	0.3426	0.3427	0.3428
	0.8	0	0.1470	0.1479	0.1469	0.1481
	1.0	0	0.0	0.0020	0.0	0.0

Table 9 Strains obtained from statically equivalent stresses from CPT

		ϵ^*_x		ϵ^*_z		ϵ^*_{xz}	
a/h	ξ	CPT	Exact	CPT	Exact	CPT	Exact
100	0.0	0.0	0.0	0.0	0.0	0.04084	0.04084
	0.2	0.19997	0.19993	−0.05990	−0.05989	0.03920	0.03920
	0.6	0.59992	0.59986	−0.17974	−0.17972	0.02614	0.02614
	1.0	0.9999	1.00000	0.29967	−0.29970	0.0	0.0
25	0.0	0.0	0.0	0.0	0.0	0.16336	0.16328
	0.2	0.19953	0.19894	−0.05844	−0.05827	0.15683	0.15676
	0.6	0.59875	0.59800	−0.17583	−0.17561	0.10455	0.10460
	1.0	0.99842	1.0005	−0.29474	−0.29537	0.0	0.0
10	0.0	0.0	0.0	0.0	0.0	0.40841	0.40705
	0.2	0.19708	0.19336	−0.05026	−0.04918	0.39207	0.39103
	0.6	0.59218	0.58739	−0.15394	−0.15254	0.26138	0.26208
	1.0	0.99013	1.00350	−0.26710	−0.27111	0.0	0.0

Conclusions

Higher-order theory based on cubic in-plane displacements and quadratic variation of the normal displacement is seen to predict accurately all quantities except the normal stresses. Statically equivalent estimates to normal stresses agree closely with the exact solution. The present theory has no provision for satisfaction of the normal stress condition at the surfaces of the plate. Lack of an expansion of displacements in a form that will satisfy all plate surface conditions remains a main hurdle in the formulation of consistent displacement-based plate models. Accurate estimation of the normal stress, using constitutive law, does not appear to be feasible. On the other hand, since the statically equivalent estimates even in CPT are good, it is expedient to pursue this direction for modeling laminated composites.

Appendix

In the theories considered for the comparative study, namely the CPT, LT, and HTPD, it is not possible to obtain all the stresses using the constitutive law; σ_x is obtained using the constitutive relation in the form $\sigma_x = E\epsilon_x$ in all the theories considered here. The shear stress is obtained using the constitutive relation $\sigma_{xz} = G\epsilon_{xz}$ in LT and HTPD, whereas in CPT it is obtained by integrating the equilibrium equation $\sigma_{x,x} + \sigma_{xz,z} = 0$. The normal stress is obtained in all three cases by integrating the second equilibrium equation. Main relations in various theories with reference to the example problem are given under. It may be noted that, in this case, $v = 0$ and all derivatives with respect to y are zero and so the equations given below contain u and W only.

Classical Plate Theory (beam theory in the present case)

$$\bar{U} = -zW,_x, \qquad \bar{W} = W$$

$$\epsilon_x = -zW,_{xx}, \qquad \epsilon_z = \epsilon_{xz} = 0$$

$$\sigma_x = E\epsilon_x$$

$$\sigma_{xz} = -\int \sigma_{x,x}\, \mathrm{d}z + A = -\frac{E}{2}[W,_{xxx}][z^2 - h^2]$$

A is evaluated using the condition $\sigma_{xz} = 0$ at $z = \pm h$.

$$\sigma_z = -\int \sigma_{xz,x}\, \mathrm{d}z + B = \frac{EW,_{xxxx}}{2} h^2 z\left[1 - \frac{z^2}{3h^2}\right]$$

B is evaluated using the condition $\sigma_z = 0$ at $z = 0$.

Governing equation:

$$EIW,_{xxxx} = 2q_0$$

S.S. end:

$$W = 0, \; W,_{xx} = 0$$

Strains ϵ_x, ϵ_{xz}, and ϵ_z from the statistically equivalent stresses are

$$\epsilon_x = \frac{1}{E}(\sigma_x - v\sigma_z), \quad \epsilon_z = \frac{1}{E}(\sigma_z - v\sigma_x), \quad \epsilon_{xz} = \frac{\sigma_{xz}}{G}$$

Levinson's Theory[7]

$$\bar{U} = -\frac{z^3}{3h^2} W,_x + \left[z - \frac{z^3}{3h^2}\right] u$$

$$\bar{W} = W(x)$$

$$\epsilon_x = -\frac{z}{3h^2} W,_{xx} + \left[z - \frac{z^3}{3h^2}\right] u,_x$$

$$\epsilon_z = 0$$

$$\epsilon_{xz} = [1 - (z/h^2)][u + W,_x]$$

$$\sigma_x = E\epsilon_x; \qquad \sigma_{xz} = \frac{E}{2(1+v)}\epsilon_{xz}$$

$$\sigma_z = -\int \sigma_{xz,x}\, \mathrm{d}z + c = \frac{E}{2(1+v)}\left[z - \frac{z^3}{3h^2}\right][u + W,_x]$$

C is determined using the condition $\sigma_z = 0$ at $z = 0$.

Governing equations:

$$\frac{2}{3}\frac{\mathrm{d}}{\mathrm{d}x}[AG(u + w,_x)] = -2Q$$

$$\frac{2}{3}AG[u + w,_x] + \frac{EI}{5}(w,_{xxx} - 4u,_{xx}) = 0$$

S.S. end:

$$w = 0; \qquad 4u,_x - w,_{xx} = 0$$

Higher-Order Theory Using Partial Deflections[18]

$$\bar{U} = -zW,_x - z[1 - (z^2/3h^2)]u$$

$$\bar{W} = W + w$$

$$\epsilon_x = -zW,_{xx} - pu,_x$$

$$\epsilon_z = 0$$

$$\epsilon_{xz} = w,_x - [1 - (z^2/h^2)]u$$

$$\sigma_x = E\epsilon_x; \qquad \sigma_{xz} = \frac{E}{2(1+v)}\epsilon_{xz}$$

$$\sigma_z = -\int \sigma_{xz,x}\, \mathrm{d}z + A$$

$$\sigma_z = z[1 - (z^2/3h^2)]u - zw,_x$$

A is determined using the condition $\sigma_z = 0$ at $z = 0$.

Governing equations are the following:

$$Bw,_{xxxx} + bu,_{xxx} = 2Q$$

$$Sw,_{xx} - su,_x = -2Q$$

$$Bu,_{xx} - Su + bw,_{xxx} + sw,_x = 0$$

S.S. end:

$$W = 0, \; W,_{xx} = 0, \; u,_x = 0, \; w = 0$$

$$B_0 = (2/3)Eh^3 \qquad S_0 = 2Gh$$

$$B = (136/315)Eh^3 \qquad S = (16/15)Gh$$

$$b = (8/15)Eh^3 \qquad S = (4/3)Gh$$

Acknowledgments

This work was supported by Aeronautics Research and Development Board, Directorate of Aeronautics, Ministry of Defence, Government of India, under the Grant No. 383.

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